Risk Sharing and Asset Allocation in Corporate Pension Schemes

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Abstract

We model the asset allocation decision of a stylized corporate defined benefit pension plan in the presence of hedgeable and unhedgeable risks. We assume that plan fiduciaries – who make the asset allocation decision – face non–linear payoffs linked to the plan's funding status because of the presence of pension insurance and a sponsoring employer who may share any shortfall or pension surplus. We find that even simple asymmetries in payoffs have large and highly persistent effects on asset allocation, while unhedgeable risks exert only a small effect. We conclude that institutional details are crucial in understanding DB pension asset allocation.

Keywords: pensions; portfolio allocation; corporate finance

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1. Introduction

Corporate defined benefit (DB) pension plans – in which employers promise members a pre–defined pension benefit from retirement – are still an extremely important part of global capital markets, with global assets and liabilities in the trillions of dollars. Understanding the underlying determinants of the asset allocation of these plans is therefore an important issue which may have much broader ramifications than the asset allocation of pension plans.

Yet existing theoretical models of the asset allocation of these plans are not able to explain why DB pension plans, especially in the United Kingdom and the United States, invest so heavily in assets which are mismatched to their liabilities. For instance, in the UK in 2008, the average DB pension plan invested 51% of its assets in equities (in the US, a comparable figure was 52%).[1] This degree of mismatching exposes their sponsors to large amounts of financial risk, with potentially destabilising consequences. Standard life–cycle models of asset allocation, or standard models of portfolio allocation such as mean–variance analysis, are inappropriate tools for understanding the asset allocation decisions of corporate pension plans because they ignore the relationship between the corporate sponsor and the pension plan. However, we argue that pure corporate finance perspectives, in turn, ignore the role played by the pension fiduciaries, who actually make plan–related decisions.

In the UK, for instance, the trustees of a scheme – who are separate from the management of the sponsoring company and who have an explicit responsibility to the members of the pension scheme – play a fiduciary role and are the ones with the ultimate decision making power over asset allocation. In the US, the sponsoring company is usually the pension fiduciary, but performs this role in accordance with strict legal guidelines. This means that it is not appropriate to assume that those who make decisions over pension

[1] The UK figure comes from the Purple Book of the Pensions Regulator (2008), which shows that equity investment in DB plans in the UK has been broadly stable at this level since at least 2006. The US figure is from Alpert *et al* (2009). Purple book data also shows that more underfunded plans in the UK invest more in equities, but little apparent relationship between plan investment strategies and sponsor financial strength.

fund policy act entirely in the interests of the shareholders of the sponsoring company, or entirely in the interests of members.

In fact, the payoffs to the members of schemes are not linked in a straightforward way to the value of the fund of assets. Members may share in scheme surplus, if it arises, and shareholders may make up any deficit. But there is an implicit asymmetry: shareholders are typically obligated to make up the full value of any deficit, if it arises, but often struggle to reclaim any surplus. For instance, in the UK, pension surpluses which arose in the late 1980's were typically spent by a combination of lower pension contributions from the sponsor and benefit improvements to members in the form of revaluation of benefits in deferment and indexation of pensions once in payment. In the US, the legal uses of surpluses in pension schemes are prescribed, and any reversion of surplus pension assets to employers is taxed at a punitive rate.

Further, in many jurisdictions, employers are required to purchase insurance to cover – often only partially – their employees against the loss of their pension benefits in the event of employer default.[2] But because premiums may not reflect the value of the insurance held by the plan, and in any case are usually paid by the sponsor rather than by the plan members, insurance may enhance the asymmetric nature of the payoffs faced by scheme members.

We argue that these features become particularly important when schemes are closed to new members and to new accruals, such as is increasingly the case in the US and the UK. The liabilities of such schemes increasingly resemble portfolios of life annuities. In this sense, they are fundamentally different from open schemes, where new generations of workers can potentially be relied upon to help sponsors bear the cost of the plan in the form of wage adjustments and future benefit changes. In closed schemes, with each year that passes, the ultimate time horizon at which the benefits become indistinguishable from a pool of annuities gets nearer. The sponsor and members effectively become

[2] Examples are the Pension Benefit Guarantee Corporation (PBGC) in the US; the Pension Protection Fund (PPF) in the UK; the Pension Benefits Guarantee Fund (PBGF) in the Canadian Province of Ontario, and *Pensionssicherungsverein auf Gegenseitigkeit* (PSVaG) in Germany.

increasingly exposed to the crystallisation of the plan risks – investment risks, longevity risks, and, for members, the risks due to the potential insolvency of the plan sponsor. Sponsors also find it increasingly difficult to reclaim any surplus arising in the plan as the plan nears maturity. Further, the fewer active members in the plan, the less the exposure of the plan to long-run inflation risks, which may affect the plan's investment strategy.

In this paper, we focus on the relationship between asymmetries in pension fund governance and the asset allocation decision made by pension fiduciaries. Although in practice there may be several asymmetries caused by the governance of pension schemes, for simplicity we incorporate only one into a dynamic, stochastic portfolio allocation model that allows for both hedgeable and unhedgeable risks facing DB pension plans. We find that this has a very significant effect on the asset allocation decisions made by the plan. Simply by allowing for the sharing of surplus and deficits in the context of an ever–more mature scheme, our model produces radically different conclusions from much prior work on occupational DB pension scheme asset allocation. In marked contrast to that work, our analysis is broadly able to explain broad features of the observed asset allocation of DB pension schemes.

Pension funds are some of the most important players in global capital markets. Therefore, if pension fund governance has a significant influence on pension fund asset allocation, it may also potentially have a profound effect on the aggregate level of risk–taking and real investment in an economy.

In the next section, we place our paper in the context of prior theoretical work on DB pension asset allocation. Further sections present our model, and derive our results. We finish with a discussion of some empirical implications of our model and a conclusion.

2. Prior work on DB pension plan asset allocation

The first work on DB pension asset allocation that uses modern financial tools dates to Sharpe (1976). He acknowledged that the payoff to pension plan members was a non–

linear function of the assets of the firm guaranteeing the promise, which he assumed included the pension scheme assets. Using options pricing methodology, he showed that under ideal conditions the pension investment policy and the pension funding policy had no effect either on the value of the corporation or on the value of the employee interest in it. His conditions are broadly equivalent to those underlying the Miller-Modigliani invariance proposition (Miller and Modigliani, 1958) , and his results can be viewed as an extension of that result to include employees as a source of financing for the business in addition to bondholders and shareholders.

Later work, by Black (1980), and then by Tepper (1981) showed that if the corporate sponsor makes good any shortfall in the pension plan, and can claim any surplus, then pension fund members bear the credit risk of their employer, but not any risk associated with the underperformance of pension assets. Here, the optimal asset allocation in the pension scheme becomes a corporate finance issue, as pointed out by Exley *et al* (1997). In the absence of other market imperfections such as bankruptcy costs and transactions costs, the effect of taxation predominates, and pension fund assets should be entirely invested in bonds.

But where employees are irrational, or offsetting wage adjustments are no longer possible, or if any surplus in the pension plan could potentially not revert entirely to the employer, neither the results of Sharpe (1976) nor the conclusions of Black (1980) and Tepper (1981) hold.

Improperly priced pension insurance can exert substantial effects on asset allocation, which none of these papers discusses. Marcus (1986) pointed out that PBGC premiums in the US were not pricing risk fairly, and Hsieh et al (1994), Coronado and Liang (2005), and Rauh (2009) have tested the extent to which PBGC insurance influences corporate pension investment policy. But there does not appear to be a theoretical model which estimates how much and under which circumstances PBGC insurance should influence plan investment.

The analysis of open DB pension plans is complex because of their inter-generational nature. These plans not only share risks between the employees and the employer, but also allow different generations of employees to share risks between themselves, potentially with great advantages. For instance, Gollier (2008) and, in the Dutch context, Cui *et al* (2009), have examined the optimal asset allocation strategy of an open pension scheme, assuming that future generations can be compelled to participate in the arrangement and that surpluses and deficits can be transferred between generations. They find that it is optimal for such schemes to take large amounts of investment risk, much larger, for instance, than standard asset allocation models might predict.[3] This is because the guarantee provided by the presence of future generations increases the ability of the scheme to bear risk, with consequent large average welfare gains for participants.

A fundamental problem with private–sector inter–generational risk sharing arrangements, as was pointed out by Allen and Gale (1997), is that future generations cannot be compelled to participate in them. Recent changes in most countries which have – or had – occupational DB pension systems illustrate this difficulty well. Plan deficits have ballooned in recent years as a result of falling interest rates, rising longevity, and poorly–performing equity markets. Employers have found themselves unable either to cut pension benefits owed to current generations or to pass on the costs of the consequent underfunding onto the next generation of workers. In Anglo–Saxon countries, where individual employers typically run their own schemes, these problems are particularly acute and many – if not most – private–sector employers have elected to close their DB pension plans. In Holland, where most schemes are industry–wide, it is easier to pass risks on to future generations, but the fundamental difficulty remains.

In complete markets, any risks borne in pension plans would be irrelevant for individual members and for the sponsor's shareholders. Both could transfer their claim in their corporate pension plan to third parties through markets – either directly or by adjusting the asset allocation elsewhere in their portfolios – and thereby attain a preferable asset

[3] In related work, Beetsma and Bovenberg (2009) examine the design and asset allocation of pension schemes, and find that mismatches between assets and liabilities of DB pension allow otherwise untraded risks to be shared between generations.

allocation at low or no cost. Individual members of DB schemes could then behave like standard life–cycle investors, as have been examined in a large academic literature (see, for instance, Cocco *et al* (2005), Koijen *at al* (2006) and Viceira (2001), amongst others).

In practice, markets are very far from complete. Individuals cannot trade their pension claims, so there are no market prices to allow them to assess the value of their claim. Indeed, assessing the value of an individual pension claim requires special knowledge and information, and is likely to be costly relative to the average size of individual claims. And even if individual pension fund members could accurately assess the extent of their exposure to the insolvency of their employer, they have no practical way to reduce it. They also cannot trade their exposure to either systematic or individual mortality risks, and few have the knowledge or capability to assess and implement their optimal exposure to interest rate risks. These factors are exactly why pensions, which involve relatively uninformed individuals, are subject to so much regulation, as pointed out by Plantin and Rochet (2007) in the context of insurance companies. Precisely because markets are so incomplete, standard life–cycle analyses actually have little or nothing to say about the asset allocation decisions of corporate DB pension plans.

No prior work on asset allocation in DB pension plans directly examines the role played by scheme governance. Pension schemes in many countries are entities distinct from the corporations which sponsor them. In the UK, for instance, the trustees of a scheme – who are separate from the management of the sponsoring company and who have an explicit responsibility to the members of the pension scheme – play a fiduciary role and are the ones with the ultimate decision making power over asset allocation.[4] In the US, the sponsoring company is usually the pension fiduciary, but performs this role in accordance with strict legal guidelines. This means that it is not appropriate to assume that those who make decisions over pension fund policy act entirely in the interests of the shareholders of the sponsoring company, or entirely in the interests of members.

[4] Cocco and Volpin (2005) present evidence which illustrates the effect which corporate insiders have on fund contributions and investment policy. The fact that some fiduciaries have responsibility to both the company and the members merely strengthens our argument that payoffs to fiduciaries are a complex function of assets and liabilities.

In such a context, there is no widely accepted answer to a set of fundamental questions facing those who make portfolio allocation decisions for pension funds. How does the scheme's institutional structure affect the asset allocation decision of the scheme? More particularly, what is the interaction between interest rate risk, equity risk and the liabilities of the scheme in the presence of substantial unhedgeable longevity risk? Should the portfolio allocation decision be different depending on the maturity of the scheme and how should it depend on today's level of funding? Finally, how sensitive to the strength of the corporate sponsor should asset allocation decisions be?

Because of the size of pension funds in many countries, and the ever–decreasing time horizon of the liabilities, these issues will become increasingly important for financial markets generally. Our paper throws light on all these questions by developing a dynamic, stochastic portfolio allocation model that takes account of the features that are important for closed pension funds, many of which make asset allocation decisions quite different from those taken by other investors, and very different from the predictions of most existing theories of asset allocation.

3. The Pension Model and Sources of Uncertainty

3.1 Overview

In contrast to previous work, we choose to analyse the investment decision using a simple but fundamental assumption: those with the responsibility for asset allocation take into account the governance framework in which the scheme operates when considering the implications of their decisions for the possible future deficit/surplus of the fund.[5] Governance issues cause those who are making the asset allocation decision to bear only a portion of the cost or benefit of the decision *ex post*. Furthermore, the proportion of the cost or benefit that they bear depends on whether the scheme is in surplus or in deficit.

[5] This does not mean that those responsible for investment decisions ignore the interests of the sponsoring company and its shareholders. In our framework, the prospect that asset out–performance can generate a fund surplus can be valued, even though most (or all) of that upside benefit may accrue to the sponsoring company.

Our main goal is to examine the effect of this governance-related factor on the appropriate allocation of funds between two broad classes of assets, bonds and equities. In our theoretical work, we choose to assume that there is only one non-linearity in pension scheme payoffs, although in reality there may be several, a point to which we return in a subsequent section. To model the fact that fiduciaries can hedge some of the liability risks they face, we incorporate a simple model of the yield curve and allow fiduciaries to choose the duration of the bonds they invest in. We also incorporate mortality risk into the liabilities, which we assume the fiduciaries cannot hedge. For each risk, we use the simplest stochastic model which has all the features we need. We abstract from virtually all other features of pension fund management by making highly stylized assumptions.

We model a closed scheme and ignore future pension contributions and the accrual of new pension liabilities, to shut off the ability of sponsors and fiduciaries to pass costs on to subsequent generations. This is not an unrealistic assumption in the UK and US, where many defined benefit corporate pension schemes are closed to new members and an increasing proportion are closed to additional accruals of pension claims of existing members. But further, we have argued that scheme governance is likely to affect asset allocation most acutely in such closed plans.

Both asset values and pension liabilities in the model are uncertain. Some of the factors that influence them are common and affect asset and liability prices (e.g., bond yields) while others are assumed to be quite distinct (e.g., longevity has a major effect upon the value of pension liabilities but we assume that it does not impact the value of any assets).

The size of the liabilities is assumed to become known at some point several years in the future. Asset allocation decisions are made dynamically at regular intervals, taking into account the level of assets and liabilities in the scheme at each point in time. When the terminal time horizon is reached, the scheme will resemble a portfolio of life annuities which have prices that reflect life expectancy and the yield on bonds at that time.

At the time horizon, the relative size of assets in the pension fund to the cost of paying pensions in full is the payoff which enters the preference function of those who decide pension fund asset allocation.

At this point, we allow for the fact that when assets exceed liabilities not all of that benefit will feed through to members (indeed only a small part might); since fiduciaries cannot be assumed to identify completely with shareholders it is therefore not appropriate that all of the excess of asset values over the cost of pensions should feed into the payoffs of the decision makers. We also allow for the fact that a corporate sponsor might make good any shortfall between asset values and pension costs, but with less than certainty. Finally, we permit the possibility of (less than full) insurance should the corporate sponsor not be able to make good a deficit in the pension scheme.

Crucial to the optimal choices is how the fiduciaries assess the costs of having a deficit at date T, and what the benefits (if any) of assets exceeding liabilities at T are. Those costs, and potential benefits, are likely to be asymmetric. In other words, the cost of assets falling 1% short of liabilities at date T is likely to be greater than the benefit of assets exceeding liabilities by 1%. Indeed, if pensioners cannot gain any of the upside benefits of asset out–performance, and if those managing the pension fund identify completely with the interests of the pensioners, there may be no value in having a surplus at all. Of course, the fiduciaries may also give weight to the interests of the sponsoring company and so attach significant value to the chance that assets exceed pension liabilities even if the entire excess may, effectively, accrue back to the sponsoring company and its shareholders.

Unless there is complete certainty that the sponsoring company will make good any shortfall at time T, and that any out–performance of assets over liabilities is of no value, then these risk preferences of those acting on behalf of members are central to the asset allocation decision. Neither of those conditions – complete certainty that the corporate sponsor will stand fully behind the scheme at time T and that if assets exceed statutory pension obligations all the excess accrues back to the company – is plausible. Since it is highly unlikely that for any DB pension fund these conditions hold, the risk preferences of the fiduciaries about the way fund assets are invested do matter.

We describe the risk preferences of the fiduciaries with a utility function that allows for risk aversion. We also allow for the payoffs that ultimately matter to the decision makers to differ from the unadjusted balance between asset values and pension costs at time *T*. If asset values fall short of pension costs the sponsoring company and/or a pension fund insurer may step in; when assets exceed pension costs those making decisions over the pension scheme may not get the full value of that because part of the gain – indeed maybe most of it – is channeled back to the sponsoring company and those returns are not valued in the same way as money flowing to scheme members.

3.2 The model in detail

Our model has three state variables which are sufficient statistics for the state of the pension fund at each point in time t – the market value of assets, A_t, the level of mortality, which is controlled by a parameter k_t, and the yield curve, controlled by the instantaneous short rate, r_t. We discuss each of these in turn, then turn to the definition of assets and liabilities at the terminal horizon and finally examine the preference function and portfolio problem of the plan fiduciaries.

Interest rates

We use a one–factor model of the yield curve to allow us to model the effect of changes in interest rates on the pension plan, but to keep the analysis as simple as possible. Therefore, the risk–neutral instantaneous short rate is assumed to follow the Vasicek model, so:

$$dr_t = a(b - r_t)dt + \sigma_r dB_t \qquad (1)$$

We assume that the rate faced by the pension fund in the market is equal to this risk neutral rate, so implicitly we assume there are some risk neutral agents in the economy who operate in the bond market and have deep enough pockets to drive bond pricing.

This implies that the conditional distribution of $r_{t+\lambda}$ given r_t is:

$$r_{t+\lambda} \mid r_t \sim N(r_t e^{-\lambda a} + b(1 - e^{-\lambda a}), \frac{\sigma_r^2}{2a}(1 - e^{-2\lambda a})) \tag{2}$$

The current shape of the yield curve that is consistent with no arbitrage can be derived from the above relation, allowing all riskless bonds to be priced. We denote by $P(i, r_t)$ the price of a zero–coupon bond that matures in i years time (in other words at time $t+i$) with a maturity value of 1, conditional on a current spot rate of r_t. Well-known bond pricing formulae are given in the appendix.

Our assumption of a one–factor model of interest rates implies that only two bonds – which we denote long and short – are necessary to permit our pension fund to achieve any sensitivity of the value of its assets to interest rates.

Mortality rates

We use the Lee–Carter (1992) model of mortality, the canonical model of stochastic mortality used in demography, which assumes that:

$$\log(m_{x,t}) = a_x + b_x k_t , \tag{3}$$

where $m_{x,t}$ is the crude death rate of individuals aged x in year t (so $m_{x,t}$ is the expected number of deaths of people aged x last birthday in year t divided by the total number aged x last birthday in that year). The state variable k_t follows a random walk with drift:

$$k_{t+1} = k_t + \mu_k + \varepsilon_t^k \text{, where } \varepsilon_t^k \sim N(0, \sigma_k^2). \tag{4}$$

The degree of uncertainty about the evolution of life expectancy is reflected by σ_k The expected rate of improvement of life expectancy over time is reflected in μ_k.

Pension assets and liabilities

We focus on the value of assets and liabilities of the scheme at a terminal time horizon, which we denote T. The value of the pension obligations at that date is L_T, and assets are given by A_T. At this point we assume that those to whom pensions are due have a remaining life that can stretch as far ahead as 35 years (we think of them as 65 year olds who have a maximum age at death of 100). The pricing of annuities, and therefore the value of pension obligations, will depend on the pattern of anticipated future mortality rates at that time. We denote the *expected* mortality rate at time T for this cohort for j periods ahead, at which point they are aged $65+j$, by

$$m(65+j, j, k_T) = E_T m_{65+j,j} \tag{5}$$

where $m_{65+j,j}$ is a function of k_{T+j}, given in (3), and $m(65+j, j, k_T)$ is a function of the random variable k_T.

We denote the present value at time T of payment of one unit at some period $T+i$ by $P(i,r_T)$. We let the value of pension liabilities at time T be equal to the price of an annuity sold at an actuarially fair rate which pays a level real amount of one per period and until death for someone aged 65 at period T, conditional on k_T. This is:

$$L_T = \sum_{i=0}^{35} P(i+1, r_T)\left(\prod_{j=0}^{i}(1-m(65+j, j, k_T))\right) \tag{6}$$

Utility function for fiduciaries

We assume that the utility function, or payoff function, for those with responsibility for the pension fund – who we call fiduciaries – depends on the funding ratio at time T. The funding ratio is A_T/L_T. But we make adjustments to the funding ratio to allow for the fact that if $A_T > L_T$ not all the benefit accrues to members. We apply a discount factor, a sort of tax rate, of t to any surplus. We also allow for a probability, p, of the sponsoring company being willing and able to make good any deficit. If the sponsoring company does not make good any deficit then an insurer covers a proportion, s, of any shortfall.

Thus, the effective stock of assets at T, which we denote A_T^* is defined as follows:

$$A_T^* = \begin{cases} \begin{cases} L_T \text{ with probability } p \\ A_T + s(L_T - A_T) \text{ with probability } (1-p) \end{cases} \text{if } A_T < L_T \\ L_T + (1-t)(A_T - L_T) \text{ if } A_T \geq L_T \end{cases} \tag{7}$$

There are two ways to interpret t. We could assume that fiduciaries care only about funds available to pay pensions and that a proportion $1 - t$ of any excess funds in the scheme flows through to members in enhanced pensions. This is the interpretation we will adopt in what follows. But fiduciaries may attach weight to the interest of the shareholders of the company and we could interpret $1 - t$ as reflecting that weight. Either way we expect $0 < t < 1$.

We use a power utility function which allows for risk aversion with respect to the adjusted, or effective, outcomes. The argument of the utility function is the ratio of adjusted assets to liabilities.

$$U(k_T, r_T, A_T) = Eu(A_T^* / L_T) \tag{8}$$

The function u is given by a standard power utility function, so that the fiduciaries are assumed to have constant risk aversion over the adjusted funding ratio. At time T, once asset price and longevity are known but before the status of the corporate sponsor is known, the expectation is over the random variable contained in the definition of A_T^* which reflects whether the sponsor bails an under–funded pension scheme out at the time horizon T or not.

The portfolio problem

We allow fiduciaries to allocate their portfolio between three assets: equities (with portfolio weight α), long bonds (with portfolio weight β) and short bonds (weight $1-\alpha-\beta$).

By long bonds we mean bonds that mature ten years after the terminal date of the problem (which have a duration close to a plausible estimate of the duration of a pension liability to pay a level real pension to a 65 year old at current rates of life expectancy). Short bonds mature after one year.

Log stock returns are denoted ε^Z. The price of the long bond is P_L and of the short bond is P_S. The equation governing the development of the assets between times *t* and *t+1* is:

$$A_{t+1} = A_t(\alpha_t e^{\varepsilon_t^Z} + \beta_t \frac{P_{t+1}^L(r_{t+1})}{P_t^L(r_t)} + (1-\alpha_t-\beta_t)\frac{1}{P_t^S(r_t)}), \qquad (9)$$

where

$$\varepsilon_t^Z \sim N(\mu_e - \tfrac{1}{2}\sigma_e^2, \sigma_e^2) \ , \ P_t^L(r_t) = P(20-t, r_t), \text{ and } P_t^S(r_t) = P(1, r_t).$$

α_t (proportion of fund invested in equities at time *t*) and β_t (proportion of fund invested in the long bond at time *t*) are decision variables, chosen optimally at each time period to maximise the expected value of the trustee payoff function in the last period, conditional on all future decisions being made optimally. These will be evaluated backwards in the usual way using standard numerical dynamic programming techniques, discussed in the appendix.

To summarise, the trustee's decision problem at time period *t* can be written as:

$$\max_{\alpha_t, \beta_t} E_t[V_{t+1}(A_{t+1}, r_{t+1}, k_{t+1}) \mid A_t, r_t, k_t], \text{ subject to:}$$

$$k_{t+1} = k_t + \mu_k + \varepsilon_t^k, \text{ where } \varepsilon_t^k \sim N(0, \sigma_k^2)$$

$$r_{t+1} = r_t e^{-a} + b(1-e^{-a}) + \varepsilon_t^r, \text{ where } \varepsilon_t^r \sim N(0, \frac{\sigma_r^2}{2a}(1-e^{-2a}))$$

$$A_{t+1} = A_t(\alpha_t e^{\varepsilon_t^Z} + \beta_t \frac{P_{t+1}^L(r_{t+1})}{P_t^L(r_t)} + (1-\alpha_t-\beta_t)\frac{1}{P_t^S(r_t)}),$$

where $\varepsilon_t^Z \sim N(\mu_e - \tfrac{1}{2}\sigma_e^2, \sigma_e^2)$, $P_t^L(r_t) = P(20-t, r_t)$, and $P_t^S(r_t) = P(1, r_t)$

In the final time period, $V(k_T, r_T, A_T) = U(k_T, r_T, A_T) = Eu(A_T^* / L_T)$, where

$$A_T^* = \begin{cases} \begin{cases} L_T \text{ with probability } p \\ A_T + s(L_T - A_T) \text{ with probability } (1-p) \end{cases} \text{if } A_T < L_T \\ L_T + (1-t)(A_T - L_T) \text{ if } A_T \geq L_T \end{cases},$$

$$L_T = \sum_{i=0}^{35} P(i+1, r_T) \left(\prod_{j=0}^{i} (1 - m(65+j, j, k_T)) \right), \text{ and}$$

$$m(65+j, j, k_T) = \exp(a_{65+j} + b_{65+j}(k_T + j\mu_k)).$$

3.3 Paramaterisation

Interest rates

For the Vasicek model of the yield curve we used: a = 0.3, b = 0.02 and σ_r = 0.01. This implies that the short term (real) interest rate has an unconditional mean of 2% and with a conditional standard deviation of 1% over one year. This is roughly calibrated to the UK case. In the base case we look at results when we start from a position with the short rate at its conditional mean.

Equity returns

For the process driving equity returns we used values broadly consistent with the properties of equity returns in the UK over the past hundred years (see Dimson, Marsh and Staunton (2008)). We set the expected annual return on equities at 6%, so the equity risk premium over the unconditional mean of long-term interest rates is 4% p.a., and the standard deviation of equity returns we set to be 20% p.a.

Mortality

As explained by Dowd *et al* (2008), modelling uncertainty in parameter estimates is crucial in order to get a reasonable range for uncertainty in mortality even relatively few years ahead. Since our dynamic programming framework does not easily permit us to model parameter uncertainty – and since this is somewhat tangential to our objective in any case – we chose to ignore parameter uncertainty in the model framework but to increase the value of σ^2_k to compensate for its omission. We first calibrated the Lee-Carter model to population data from 1900-2006 for English and Welsh males over the age of 55, incorporating mortality uncertainty. Using a simulation-based approach, we found that the width of a 95% confidence interval for estimates of life expectancy at a 10-year time horizon was approximately 5.6 years (without parameter uncertainty, the 95% confidence interval was approximately 2 years wide). Choosing a value of $\sigma_k = 2.5$ in our dynamic programming framework produced a similar level of uncertainty in mortality estimates at a 10-year time horizon. We therefore considered this to be our base level of mortality uncertainty. We also examined the sensitivity of our results to this assumption. Details are given in the technical appendix.

Fiduciary preferences

In the base case we set the fiduciaries' risk aversion parameter, γ, equal to 5; this represents a substantial degree of risk aversion that we assume fiduciaries exhibit over the funding ratio. We examined the sensitivity of our results to this assumption.

For values of p, s and t we examine three different cases. First we briefly discuss the case where the pension scheme bears no risk, which we call case 0. We then examine two cases in detail, first where the pension scheme faces the entire investment risk itself (case 1), and then a more realistic situation where the sponsor shares in the upside of the scheme's returns, but also bails out the scheme with a certain probability if assets are less than liabilities, and there is a pension insurer to potentially cover members if the sponsor fails (case 2).

Case 0: No sponsor default, surplus and deficits revert to sponsor ($s = t = p = 1$).

With these values of *s*, *t* and *p*, the pension fund is entirely a part of the sponsoring company. The sponsoring company makes up the deficit to full funding and takes all the surplus if any arises. There is no chance that the sponsoring company will default. Figure 1 shows the adjusted value of the funding ratio (vertical axis) against the unadjusted value.

It is clear that our framework provides no optimal asset allocation in this case: there is no reward and no cost to the fiduciaries of following any one asset allocation plan rather than any other. In all cases, regardless of investment policy, the liabilities of the fund are fully met and hence the duties of the fiduciaries are fully discharged.

This case is examined by Exley, Mehta and Smith (1997). They argue that the asset allocation of the pension fund is irrelevant because shareholders can alter their own portfolios to take account of any asset allocation in the pension fund, though there may be tax factors (which we do not model) that mean there is an advantage to holding bonds inside the pension fund rather than outside it. This reflects arguments first set out in earlier papers by Black (1980) and Traynor (1981).

In practice, it is extremely unlikely that $s = t = p = 1$. As we have discussed, it is routine in the UK that any surplus in a pension fund is to some extent shared between members and the plan sponsor in the form of benefit improvements to existing members. In the US, sponsor access to surplus is severely curtailed by law (reversions of pension surpluses to plan sponsors are taxed at a punitive rate, and the legal uses of pension surplus other than reversion are clearly laid out in law). Furthermore, no company in either country has such a strong covenant that the chance of default is zero at every point in the future. We therefore turn to other cases.

Case 1: No pension fund sponsor ($s = t = p = 0$).

This is the other end of the spectrum from case 0. Here, the pension fund stands entirely alone without any recourse either to a public pension insurer ($s = 0$) and without any possibility of being bailed out by the corporate sponsor (p = 0). In addition, all surplus earned by the fund is kept by the fund, so $t = 0$. Figure 1 shows the adjusted value of the funding ratio (vertical axis) against the unadjusted value.

Insert Figure 1 here.

Brennan and Xia (2000) provide analytical solutions to a continuous-time model in which an agent dynamically chooses an investment policy in stocks and bonds to maximise the expected CRRA utility of terminal wealth, in the presence of stochastic interest rates. Except for the fact that they assume continuous trading and that they do not allow for unhedgeable liabilities, their model is directly comparable to our case 1. Consistent with Merton (1971), they find that investors optimally hold two mutual funds: the myopic mean-variance efficient portfolio, and a portfolio to hedge the effects of changes in interest rates. We adapt their model to our context by ignoring unhedgeable mortality risk and assuming that our liabilities can be perfectly hedged by a L-period zero-coupon bond. Since we assume that the stock price is uncorrelated with bond returns, the mean-variance efficient portfolio consists of stocks and the short bond only, and the hedge portfolio is made up only of the long bond and the short bond. The benchmark optimal proportion in equities in complete markets would then be given by:

$$\alpha = \frac{\mu_e - r_t}{\gamma\sigma_e^2}, \tag{10}$$

(the classic Merton (1969) formula), while the proportion allocated to the long bond would be given by:

$$\beta = (1 - \frac{1}{\gamma}) \frac{1 - e^{-a(L-t)}}{1 - e^{-a(20-t)}}. \tag{11}$$

In our numerical results, we assume that $L = 20$, and therefore (11) simplifies to

$$\beta = (1 - \frac{1}{\gamma}). \tag{12}$$

Case 2: Asymmetric sharing of surpluses and deficits, sponsor default ($s = 0.7$, $t = 0.5$, $p = 0.5$)

In this case, the pension insurer makes good 70% of the deficit should the corporate sponsor be unable to make good any deficit. There is a 50% chance that the sponsor will be willing and able to make up a deficit. If there is a surplus, 50% of the surplus reverts to the sponsor (or at least does not get reflected in the payoffs that matter to the scheme fiduciaries). Figure 1 shows the expected adjusted funding ratio plotted against the raw funding ratio.

Because of the sources of market incompleteness, there is no analytical solution to either case 1 or case 2 of our problem and we solve the model numerically. This is challenging, especially in case 2, because of the discontinuities and non–concavities in the problem. Details of our solution technique are given in the appendix.

4. Results

In each case, we first focus on the split between stocks and bonds, starting in the last time period before the terminal date and moving backwards. We then examine the composition of the bond portfolio. Finally, we test the sensitivity of our results to different assumptions, including the level of mortality uncertainty. For case 1, we make comparisons with the theoretical benchmark portfolio where appropriate.

Case 1: No pension fund sponsor ($s = t = p = 0$).

Table 1 presents the equity-bond split for both the numerical model and for the appropriate benchmark results. In case 1, for both theoretical and numerical results, the investment strategy does not depend on the expected funding ratio of the plan and so the results shown apply to all funding ratios. Table 2 shows the duration of the bond portfolio. For the theoretical benchmark, which makes no allowance for unhedgeable risk, we assume that the liabilities are perfectly matched by a zero-coupon bond which matures ten years after the time horizon.

Insert Tables 1 and 2 here.

The mix between equities and bonds

At our base case value of mortality uncertainty (σ_k = 2.5) our model agrees exactly with the theoretical benchmark (and with Merton's (1969) model of asset allocation). The fund invests a constant proportion of its assets in equities regardless of the expected funding ratio at the terminal date. Furthermore, again consistent with the theoretical benchmark, the length of the time horizon has no effect on the optimal split between equities and bonds. So in case 1, the optimal proportion of the portfolio invested in stocks and bonds is independent of both the time horizon and the funding ratio.

Changing the level of mortality uncertainty has no effect on the optimal portfolio split between equities and bonds, as pointed out by Merton (1971). The intuition is simple. Each period, dynamic investors hold assets which hedge the portfolio they expect they will want to hold in the next period. Unhedgeable changes in mortality, which alter the distribution of the funding ratio in the next period, could therefore influence this hedging portfolio. But if investors would hold the same assets next period for every funding level – and therefore for every realisation of mortality uncertainty – then unhedgeable mortality uncertainty will not change their portfolio holdings this period.

Finally, we examine the sensitivity of our results if we change the equity risk premium, and the level of risk aversion. Our numerical results show that the proportion of the portfolio invested in equities is linear in the equity risk premium and the inverse of the fiduciaries' coefficient of relative risk aversion, exactly consistent with the benchmark.

The composition of the bond portfolio

Table 2 shows that our model predicts similar bond investment strategies to the theoretical benchmark, at all time horizons, although there are some differences. In both cases, the optimal strategy is to hedge the liabilities of the pension fund by investing the bond portfolio primarily in long-dated bonds, although the optimal bond portfolios in our numerical model have somewhat shorter durations than the benchmark model.

Insert Table 2 here.

The difference between our optimal portfolios and the theoretical benchmark is partly explained by our use of annual time periods. We show in the appendix that the pure expectations hypothesis holds in the Vasicek model for very short holding periods if the market price of interest rate risk is zero, as we assume. But as the holding period lengthens, a strategy of rolling over short bonds produces a very slightly higher expected return than holding one longer bond over the same period. For holding periods of one year, the effect is small, but noticeable. In our numerical results, the fiduciaries take advantage of this by investing the portfolio slightly shorter than the term of the liabilities.

As in the case of the split between equities and bonds, changing the degree of mortality uncertainty has no effect on the optimal duration of the bond portfolio, and for the same reason. Reducing the equity risk premium reduces the optimal duration of the bond portfolio slightly in both the theoretical and numerical models, while increasing the equity risk premium lengthens it slightly. Reducing the coefficient of risk aversion decreases the duration of the bond portfolio in our numerical results slightly, and making it larger increases it, because of the convexity effects we have discussed. The benchmark bond portfolio is independent of the of the level of risk aversion because the pure expectations hypothesis holds and funds consequently fully hedge their liabilities at all levels of risk aversion.

Case 2: Asymmetric sharing of surpluses and deficits, sponsor default ($t = 0.5$, $s = 0.7$, $p = 0.5$)

Case 1 provides a very useful benchmark for where there is an asymmetry between the treatment of surpluses and deficits in the fund. In case 2, we examine a situation where, if the fund is in surplus at the time horizon, 50% of the surplus is shared between members and the sponsor. If the fund is in deficit, we assume that there is a 50% chance that the sponsor will make good the entire deficit by making contributions to the fund. If the sponsor defaults, a pension insurer will make good 70% of the deficit.

In all tables, we show results for different levels of the expected funding ratio at the terminal date when we discount liabilities on the current bond yield curve, when the instantaneous short rate equals its long run mean. (Since mortality risk is unhedgeable, the ultimate funding ratio is not known and must be estimated. We take simple expectations of the ultimate funding ratio over the probability distribution of the mortality risk, assuming that the fund is invested in matching bonds.)

The mix between equities and bonds

We first focus on the proportion of the fund invested in equities, shown in Table 3. Using our base case parameters, the optimal strategy at time $T - 1$, shown in the last column, is to invest the entire fund in equities, provided the fund is less than 120% funded. Above this funding level, there is a sharp drop in the proportion of equities, with about 1/3 of the fund invested in equities by the time the 150% funding level is reached. It is clear that the combination of the pension insurer and the sponsor have radically altered the optimal investment strategy of the scheme.

As the time horizon becomes more remote, equity investment starts falling below 100% at a lower funding level, but falls less rapidly as the funding level improves. Ten years before the terminal date, the fund invests everything in equities provided the funding ratio is below 90%. Above this level, bond investment increases slowly. By the time the fund is 150% funded, around 45% of the assets are invested in equities. At this point, changes in investment policy from year to year are very small, indicating that the investment policy is now almost independent of the time horizon. But the change in the payoff functions between case 1 and case 2 has caused the level of equity investment to be dramatically higher in all time periods and at all funding levels. The kink causes a significant and permanent increase in the optimal amount of risk that pension plan fiduciaries choose to take on in their portfolios.

Insert Table 3 here.

Varying the level of risk aversion and changing the equity risk premium

To explain why the change in payoff functions has such a large effect, we first illustrate how sensitive the optimal investment strategy in the final period is to the assumed level of the equity risk premium and the assumed level of risk aversion of the fiduciaries. Results are shown in Table 4.

In the base case, we assumed an equity risk premium of 4% p.a. If we reduce the equity risk premium to 3% p.a., then, as we would expect, equity investment by the plan falls. If the funding level in the penultimate period is near the full funding level – where the kink in the payoff function is – then the optimal investment strategy is to invest the entire fund in equities. As the funding level moves away from the kink, in either direction, then the investment strategy becomes rapidly more conservative. Below the kink, as the funding level declines further, the investment strategy once again becomes riskier, reaching 100% in equities again at some point. At high funding levels, the optimal investment in equities declines, eventually approaching the same constant level that was found in case 1 at very high funding levels. Thus, there is a "hump" in the optimal proportion of the fund invested in equities.

Insert Table 4 here.

For levels of the equity risk premium above 3% p.a., the proportion in equities increases where it is not already 100%. Where the assumed equity risk premium is sufficiently high, such as in the base case, the scheme never invests less than the full amount of the fund in equities at low funding levels, and so the kink is not apparent.

A similar “hump” in the equity investment of the scheme appears when the assumed level of risk aversion of the fiduciaries is increased. Decreasing the level of risk aversion increases the optimal investment in equities across the board, as expected.

There are two main factors causing this unusual non-monotonic pattern of equity investment in the penultimate period.

Firstly, the sharing of surpluses and deficits relative to the liabilities creates a gearing effect which increases the proportion of the fund invested in equities at low funding levels. For instance, a symmetrical sharing of deficits and surpluses between sponsor and trustees would be the equivalent of a proportional tax of 50% of the scheme’s assets plus a lump sum transfer of 50% of the scheme’s liabilities at the terminal date. On its own, the proportional tax would preserve the case 1 result of a constant proportion in stocks and bonds. The effect of the lump sum transfer, implicitly invested in bonds perfectly matched to the liabilities, is to increase the optimal level of financial risk taken on at lower levels of funding. This effect is exactly analogous to the gearing induced by riskless labour income in Merton’s (1969) model of lifetime consumption and investment.

Secondly, asymmetric payoffs around the point of full funding increase the incentives of the fiduciaries to take on extra risk. As the funding level moves away from the kink, the effect of the asymmetry in payoffs falls because the funding level is less likely to end up near the kink in payoffs. In our case 2, the probability that the corporate sponsor will fully make good any shortfall in assets over liabilities further increases the asymmetry in payoffs. The incentives of fiduciaries to take investment risks by raising the proportion of the fund invested in equities around the point of full funding rise in consequence.

Neither of these effects depends on the precise shape of the preference function of the fiduciaries. Provided that the fiduciaries prefer higher funding levels to lower ones, and are risk averse, the gearing effect and the asymmetry of payoffs will increase the risk taken on by trustees in their investment portfolios.

As the time horizon becomes more remote, the "hump" flattens and eventually disapperars. Ten years before the terminal date, the optimal fraction of the fund invested in equities declines smoothly as the funding level increases, although for low levels of the equity premium or high levels of risk aversion, equity investment is lower than in the base case at all funding levels where it is not 100%.

The composition of the bond portfolio

The change in the payoff function in case 2 affects the make-up of the fund's optimal bond portfolio as dramatically as it changes the split between equity and bonds. In case 1, we found that the duration of the bond portfolio was close to the duration of the liabilities, that it did not depend on the funding ratio and that it increased as the time horizon became more distant.

Case 2 results are shown in Table 5. In the penultimate time period, the fund invests virtually nothing in bonds unless the funding ratio is greater than 110%. Above this point, the influence of the kink falls rapidly, and the scheme consequently invests in a conservative mixture of equities and bonds. The duration of the bonds is quite long, around 9 years, although a little shorter than in the equivalent period in case 1.

Insert Table 5 here.

As the time horizon recedes, however, the duration of the bond portfolio falls, rather than rises, at low funding levels, in marked contrast with case 1. In case 2, the more distant the time horizon, the more the scheme invests in short-dated, rather than long-dated bonds, and rolls over the returns. This strategy is riskier than matching the interest rate exposure of the liabilities, because the scheme is exposed to reinvestment risk, but it earns the scheme a slightly higher expected rate of return as a result of convexity and our assumed one-year holding period. By ten years before the ultimate time horizon, virtually all the bond holdings of the fund are in short bonds at funding levels less than 150%. Above this point, as the funding level increases, the optimal bond portfolio will tend towards the safe long-dated bond portfolio. Just as in the equity portfolio, the kink encourages the scheme to follow a risky – but potentially rewarding – strategy in its bond portfolio. These results are shown in Table 5.

A key driver of the bond investment strategy in case 2 is the equity investment of the fund, and the expected return on this investment. These affect how relevant the kink is likely to be for the fund as the terminal date approaches, which in turn has a large influence on the degree of mismatching in the bond portfolio. This can be illustrated by showing how the bond investment strategy of the fund changes as the equity risk premium and the fiduciaries' coefficient of relative risk aversion change. The results are presented in Table 6.

Insert Table 6 here.

For low levels of the equity risk premium, if the fund is overfunded, the chance of the kink being relevant is higher than if the equity risk premium is higher. Therefore, the optimal duration of the bond portfolio is likely to be lower. As the time horizon recedes, the influence of the kink expands, and the optimal degree of mismatching increases.

Changing the coefficient of risk aversion has a similar effect on the bond portfolio, but through two channels. Firstly, when trustees are more risk averse, they hold bond portfolios with longer durations at all funding levels. This is because increasing the level of risk aversion alters the perceived risk-reward trade-off associated with mismatching the duration of the liabilities. Secondly, changing the risk aversion of the trustees changes the proportion of the portfolio that the trustees invest in equities. As in the case of changing the equity risk premium, this changes the distribution of the ultimate funding ratio, and, in particular, the probability that the kink will come into play.

Varying levels of mortality uncertainty

We now examine whether uncertainty about longevity has any effect on the investment policy of the fund when there is a kink in the payoff function. Unlike case 1, we find that increased mortality uncertainty increases the optimal proportion of the fund invested in equities, but that the effect is very small, as can be seen in Table 7.

Insert Table 7 here.

The effects are driven by the dependence of the investment strategy on the funding ratio. Unlike in case 1, the investment strategy in each period now depends on the funding ratio. Therefore, changing the distribution of the funding ratio in any period – for instance by introducing unhedgeable mortality uncertainty – could influence the portfolio that dynamic investors will choose one period before. In this way, the introduction of unhedgeable mortality uncertainty may affect the investment strategy in each period. However, our numerical results show that even ten years before the ultimate time horizon, the effects of large mortality uncertainty on the optimal policy are very small.

Monte Carlo simulations

In the previous section we focused on the optimal portfolio rule – the relation between the expected future funding ratio at a point in time and the best choice between equities, short dated bonds and long dated bonds. Here we analyse how the distribution of outcomes can evolve over time in order to illustrate the individual and aggregate consequences of the behavioural response to the kink in payoffs. We perform 10,000 Monte Carlo simulations, each of which uses a series of random numbers to generate a specific path for the stochastic variables that create uncertainty for fiduciaries – equity values, mortality rates and bond yields. We describe how the outcomes for portfolio allocation and funding status spread out if we start from a given expected funding position and values for bond yields. We look at the distribution of the funding position of pension funds that initially have 80% funding ratios; longevity risk is set at our base case level and bond yields are initially at their unconditional mean. The results are shown in Table 8.

In case 1, where the fund bears all of its own risk, half of the time funds will have ultimate funding ratios between 80% and 94%. Introducing a sponsor and a pension fund

insurer, the resulting asymmetry in payoffs means that funds whose fiduciaries follow the optimal investment strategy will have much more variable ultimate funding ratios. For instance, only 50% of funds will be between 66% and 150% funded at the terminal date. Other agents in the economy – sponsors, the pension insurer, and pension fund members – will have to bear the consequences of this higher aggregate risk taking.
The Monte Carlo results reveal that common features in scheme governance and institutions – either in the form of an explicit insurance scheme or because a corporate sponsor has a substantial probability of making good any shortfall – create incentives for fiduciaries to make portfolio allocation decisions that generate very significant variability about future outcomes. Some agent(s) in the economy – sponsors, the pension insurer, or pension fund members – will have to bear the consequences of this variability. The less strong the initial funding position of the scheme, and the greater the extent of the kink in payoffs, the greater this variability becomes.

Insert Table 8 here.

5. Empirical implications

In previous sections we have shown in a simplified model that even a single asymmetry between the treatment of surpluses and deficits in a pension scheme is a key driver of pension investment policy. Before pointing out areas in the empirical literature on pension asset allocation where our analysis could provide additional insights, we examine some issues which may complicate the empirical application of our model.

Firstly, there may be more than one non-linearity in pension scheme cash flows. For instance, the level of benefits covered by pension insurance may be significantly below the level of benefits promised by the sponsor. In this case, for funding levels between the level covered by the insurer and full funding, members could benefit entirely from any changes in the funding level of the plan. Above full funding, members and sponsors may share the surplus, and below the level covered by the pension insurer, the assets held by the plan would be irrelevant. Such a situation is shown in a stylistic way for sponsors of different strengths in Figure 2. Each kink would influence the optimal asset allocation of the plan.

Secondly, funding pension plans is at least partly endogenous. Even in the US, where pension funding is tightly regulated, pension fund sponsors have some degree of discretion, and in the UK, this discretion is quite significant. If there are large numbers of active members in a plan, sponsors can reduce contributions in order to reclaim plan surplus, even if the plan is not fully funded. Further, the greater the member share of any surplus, the more likely sponsors will be to try and reduce contributions when the plan approaches full funding.

Finally, because wind-up rules on default may have a different impact on different classes of member, and because individual members may receive different levels of pension insurance coverage depending on their level of benefits, payoffs may not be the same for different members of the plan. Understanding the true payoff function of the fiduciaries may be challenging.

We now discuss the additional insights provided by our model to the existing empirical literature on asset allocation in corporate pension plans.

The sharing of surplus between members and the sponsor

In both the US and the UK it is difficult for pension fund sponsors to reclaim surplus from their pension schemes if there are no active members. However, where pension schemes have substantial numbers of active members, sponsors can reclaim surplus by reducing their contributions.

The lower the proportion of surplus that members can retain, the less convex the payoffs to fiduciaries and the safer the investment policy they would wish to follow. We would therefore predict that the greater the proportion of active members, the *less* risky the investment strategy of the fund should be for schemes which are not too underfunded.[6]

In closed plans, we would predict that the greater the extent that surplus is shared between members and sponsors, the greater the investment risk taken on by the plan, provided the sponsor is sufficiently strong. The more weight the fiduciaries attach to the interests of members, the stronger this effect would be. It would be difficult to measure member expectations of future shares in surplus, and no work that we know of has attempted to investigate this question.

The strength of the pension fund sponsor

Both Rauh (2009) and Coronado and Liang (2005) find that firms which are facing bankruptcy appear to invest more of their pension assets in bonds than stronger firms. Rauh (2009) in particular interprets this as evidence that shareholders of weak firms are more concerned about the possibility that mandatory pension contributions required under US law will drive the firm into bankruptcy than they benefit from the ability to transfer risk to bondholders or the PBGC.

Figure 2 shows the stylised expected adjusted funding ratio faced by fiduciaries of a plan for a stronger and a weaker sponsor where the pension insurer insures 70% of the benefits, and surplus above full funding is shared equally between sponsors and members if the sponsor survives, and is taken entirely by the members if the sponsor fails.

For plans which have a funding level around where pension insurance ends, the weaker the sponsor, the greater the convexity at the lower kink and the stronger the incentive to pass risk on to the PBGC by increasing investment risk in the plan. But if the plan is near full funding, then the weaker the sponsor, the less convex the payoffs to fiduciaries and the *less* risky the optimal investment strategy. Our model therefore suggests that the

[6] This may offset the incentive effect found in theoretical work by Sundaresan and Zapatero (1997) and Lucas and Zeldes (2006), who posit that firms may invest in risky assets such as equities to hedge against the effect of salary increases on the value of benefits of active members.

result found by Rauh (2009) may hold even in jurisdictions which do not have strict contribution requirements, such as the UK.

The time horizon effects we found in our theoretical analysis may strengthen this effect. Weaker sponsors have shorter expected lives, which reduces the influence of kinks in payoff functions far from the current funding level of the plan. For instance, while the lower kink in Figure 2 may influence the investment decision of a fully funded plan when the sponsor is strong, it will have much smaller impact when the sponsor is weaker. Similarly, if plans are very underfunded, a weakening of the sponsor may actually lessen the impact of the lower kink and lessen the degree of risk taking in the plan. So our model suggests that well-funded and very poorly funded plans with weak sponsors may have a larger incentive to invest assets more safely than similar plans with stronger sponsors.

The relationship between pension funding and asset allocation

Another issue of interest is the relationship between pension scheme funding and pension risk taking. Previous empirical work on this issue has reached conflicting conclusions. For instance, Bodie *at al* (1987) and Rauh (2009) find a negative, although not necessarily statistically significant, correlation between risk-taking and scheme funding, while Petersen (1996) finds a positive correlation. In the UK, Cocco and Volpin (2005) find a statistically significant negative correlation. One empirical prediction of our model is that the pension plans of weaker sponsors should have a stronger negative relationship between funding and investment risk than equivalent plans with stronger sponsors, although the strength of the relationship for stronger firms would depend on the extent to which surpluses are shared between members and sponsors.

6. Conclusions

Prior work on the asset allocation of corporate DB pension schemes has ignored some important features of the governance of these schemes. We have shown that the asset allocation decisions made by rational, risk–averse fiduciaries are extremely sensitive even to simple asymmetries in payoffs. Three significant sources of such asymmetries in corporate DB pension plans are the possibility of the corporate sponsor bailing out a fund, the presence of an insurer and the possibility that only part of upside benefits of asset out–performance relative to scheme liabilities accrue to the scheme members provide.

We find that even slight asymmetries exert substantial effects on asset allocation, and that the greater the degree of asymmetry, the greater the size of the effect. We also find that asymmetries change the optimal asset allocation many years before the terminal horizon, even if investment decisions are revisited frequently in a dynamic context, so the effect is highly persistent over time. Provided that the payoff curve to fiduciaries is convex, our results show that any asymmetry will increase the equity investment of pension schemes. Hence, if the ultimate asymmetry in payoffs is many years in the future, and schemes revisit their asset allocation decisions frequently, rational, forward–looking fiduciaries will invest much more heavily in equities than they would if they bore both the upside benefits and the downside costs of mismatches between their assets and their liabilities. Increased equity investment results in substantial increased variability in investment outcomes.

In our framework, participants cannot hedge their positions fully by trading in financial markets because of market incompleteness, which in our analysis is provided by unhedgeable longevity risk. We find that while the presence of unhedgeable longevity risk has only a small effect on asset allocation, it serves to further increase the optimal allocation to riskier assets. The effect is more pronounced the greater is the degree of asymmetries in payoffs. Our results suggest that while unhedgeable background risk is an important factor affecting the optimal asset allocation of DB pension schemes, it is not significant enough to explain the asset allocations we observe in practice.

Our analysis suggests that careful attention needs to be paid to the governance framework in which investment decisions are made in pension plans, especially if these plans are moving towards final closure, and pension fund sponsors have no way of reclaiming the cost of poor investment outcomes from future generations of workers. If sponsors in this situation have imperfect control over the asset allocation of pension plans, then the governance framework, which was designed to give workers greater protection from investment and other risks, could actually be creating substantially more risky portfolios, and hence a much greater level of aggregate risk taking, than would otherwise be optimal. This could be the case many years before any risks are due to crystallise. Ironically, the stronger the sponsor is perceived to be, or the more complete the insurance protection offered, the greater the effect and the greater the level of consequent inefficient excess risk taking.

Our results suggest that institutional factors may be one of the prime drivers of pension fund investment strategies, and that the agency costs of the governance structure of pension funds might be much higher than have been realised. The current parlous state of pension funding in the US and UK, and the consequent financial instability of pension fund sponsors, may be one of the consequences.

7. References

Allen, Franklin and Douglas Gale, (1997), "Financial Markets, Intermediaries and Intertemporal Smoothing", *Journal of Political Economy,* 105:3, 523–546.

Alpert, Steve, Charlie Clark, Debra Nice and Alan Parikh, (2009), *How does your retirement program stack up? – 2009*, Mercer (US) Inc, New York, NY.

Beetsma, Roel W.M.J and A. Lans Bovenberg, (2009), "Pensions and Intergenerational Risk-sharing in General Equilibrium," *Economica,* London School of Economics and Political Science, 76(302): 364-386.

Black, F. (1980). "The tax consequences of long run pensions policy", *Financial Analysts Journal* 36: 21–8.

Bodie, Zvi, J.O. Light, R. Morck and R.A. Taggart, (1987), "Funding and asset allocation in corporate pension plans: an empirical investigation", in Z. Bodie, J. Shoven and D. Wise (eds), *Issues in Pension Economics*, University of Chicago Press, Chicago, IL.

Brennan, Michael J. and Yihong Xia, (2000), “Stochastic interest rates and the bond-stock mix”, *European Finance Review,* (4): 197-210.

Carroll, Christopher D., (1997), “Buffer-stock saving and the Life Cycle / Permanent Income Hypothesis”, *Quarterly Journal of Economics,* CXII(1): 1-56.

Cocco, Joâo Francisco, Francisco Gomes and Pascal Maenhout, (2005), “Consumption and portfolio choice over the life cycle”, *Review of Financial Studies,* 18, 492–533.

Cocco, Joâo Francisco and Paolo Volpin, (2005), "The Corporate Governance of Defined–Benefit Pension Plans: Evidence from the United Kingdom," CEPR Discussion Papers 4932, C.E.P.R. Discussion Paper

Coronado, Julia and Nellie Liang, (2005), “The influence of PBGC Insurance on Pension Fund Finances”, Pension Research Council working paper 2005-10.

Cui, Jiajia, Frank de Jong and Eduard Ponds, (2009), “Intergenerational risk sharing within funded pension schemes”, Working Paper, All Pensions Group, Tilburg University.

Dimson, Ellroy., Paul Marsh and Mike Staunton, (2008) *Triumph of the Optimists: 101 years of investment returns*, Princeton University Press, Princeton, NJ.

Dowd, Kevin, Andrew J.G. Cairns, David Blake, Guy D. Coughlan, David Epstein, and Marwa Khalaf-Allah, (2008), *Backtesting Stochastic Mortality Models: An Ex-Post Evaluation of Multi-Period-Ahead Density Forecasts.* Pensions Institute Working Paper, Cass Business School, City University, London.

Exley, C. Jon, Shyam J.B. Mehta, and Andrew D. Smith. (1997). “The financial theory of defined benefit pension schemes”, *British Actuarial Journal* 3 (Part IV): 213–383.

Gollier, Chistian, (2008), “Intergenerational risk sharing and risk–taking of a pension fund”, *Journal of Public Economics* 92 (2008) 1463–1485.

Koijen, Ralph S.J., Theo E. Nijman and B.J.M. Werker, (2006), “When can life–cycle investors benefit from time varying bond risk premia ?”, Netspar Discussion Paper, Tilburg University

Lee, Ron D., and Lawrence R. Carter, (1992), “Modeling and forecasting U.S. mortality”. *Journal of the American Statistical Association*, 87(419), 659–671.

Lucas, Deborah and Stephen P. Zeldes, (2006), “Valuing and Hedging Defined Benefit Pension Obligations: The Role of Stocks Revisited”, working paper, Columbia University.

Marcus, Alan J., (1986), "Spinoff/Terminations And The Value Of Pension Insurance,"Journal of Finance, 1985, v40(3), 911–924

McCarthy, David G. and Anthony Neuberger, (2005), “The Pension Protection Fund”, *Fiscal Studies,* Vol. 26, No. 2, June pp. 139–167

Merton, Robert C., (1969), “Lifetime portfolio selection under uncertainty: the continuous-time case”, *Review of Economics and Statistics*, 51: 247-257.

Merton, Robert C., (1971), “Optimum Consumption and Portfolio Rules in a Continuous-Time Model”, *Journal of Economic Theory*, 3: 373-413.

Miller, Merton and Franco Modigliani, (1958), "The Cost of Capital, Corporation Finance and the Theory of Investment". *American Economic Review,* 48(3): 261–297.

Petersen, M., (1996), “Allocating assets and discounting cash flows: Pension plan finance”, in P.A. Fernandez, J.A. Turner and R.P. Hinz (eds), *Pensions, Savings and Capital Markets.* Washington D.C.: U.S. Department of Labor.

Plantin, Guillaume, and Jean–Charles Rochet, (2007), “When Insurers Go Bust: An Economic Analysis of the Role and Design of Prudential Regulation”, Princeton University Press.

Rauh, Joshua, (2009), “Risk shifting versus risk management: Investment policy in corporate pension plans”, *Review of Financial Studies*, 22(7):2687-2733.

Sharpe, William F., (1976), "Corporate pension funding policy," *Journal of Financial Economics*, 3(3): 183–193.

Sundaresan, S., and F. Zapatero, (1997), “Valuation, Optimal Asset Allocation and Retirement Incentives of Pension Plans”, *Review of Financial Studies*, 10: 631-60.

Tepper, Irwin, (1981), “Taxation and Corporate Pension Policy”, *Journal of Finance*, 36(1): 1-13.

The Pensions Regulator, (2008), *The Purple Book*, *2008*, The Pensions Regulator, Brighton.

Treynor, Jack, (1977), “The principles of corporate pension finance”, *Journal of Finance* 32: 627–38.

Viceira, Luis, (2001), “Optimal portfolio choice for long horizon investors with non–tradable labor income”, *Journal of Finance,* 56, 433–470.

Wang, Duolau and Pengjun Lu, (2005), “Modelling and forecasting mortality distributions in England and Wales using the Lee-Carter model”, *Journal of Applied Statistics*, 32(9): 873-886.

8. Technical appendix

Calibrating mortality parameters

To incorporate parameter uncertainty into the Lee-Carter model, we followed Wang and Lu (2005) and altered the two-step fitting procedure originally used by Lee and Carter. Rather than using singular value decomposition in the first stage, we used maximum likelihood estimation, from which estimates of a_x, b_x and k_t, as well as an estimate of the variance-covariance matrix of b_x, could be obtained.[7] In the second stage of the estimation, we fit an ARIMA(0,1,1) process to the values of k_t obtained from the first stage, again using maximum likelihood to estimate the variance-covariance matrix of the parameter estimates.

We took 10,000 draws from the estimated parameter distributions of b_x and the parameters governing k_t. For each draw, we simulated a 10-year path of the future development of k_t, and calculated the implied life expectancy for a male aged 65 in ten years' time conditional on the simulated value of k_{t+10} and that draw's values of b_x and the parameters of the process governing k_t, assuming all future shocks to k_t after time *t+10* were exactly equal to zero. This produced an estimate of the uncertainty in *projected* life expectancy in ten years which incorporated the effects of parameter uncertainty.

Using population data from 1900-2006 for English and Welsh males over the age of 55, we found that the standard deviation of the estimate of life expectancy of a 65-year old in ten years time was approximately 1.4 years, leading to a 95% confidence interval which had a width of 5.6 years. (Without parameter uncertainty, the width of the confidence interval is closer to 2 years).

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[7] We assumed that the number of deaths at each age in each year has a Poisson distribution with parameter equal to $m_{x,t}n_{x,t}$, where $n_{x,t}$ is the number of individuals aged *x* in year *t*. Since we were using population data, the Poisson approximation was not restrictive.

We then ran a set of simulations assuming that the parameters governing the process k_t were fixed and known, but for different values of σ^2_k , the variance of the innovations of k_t . Setting σ_k = 2.5 produced a similar level of uncertainty in mortality estimates at a 10-year time horizon to the maximum likelihood estimates incorporating parameter uncertainty. We therefore considered this to be our base level of mortality uncertainty. We also examined the sensitivity of our results to this assumption.

Vasicek model of interest rates and the pure expectations hypothesis

The Vasicek model implies that bonds can be prices by the following formula:

$$P(i,r_t)=C(i)e^{-B(i)r_t}\,,\,B(s)=\tfrac{1}{a}(1-e^{-as})\,, \qquad \text{(A1)}$$

where

$$\log(C(s))=(b-\frac{\sigma_r^2}{2a^2})(B(s)-s)-\frac{\sigma_r^2}{2a}B(s)^2\,. \qquad \text{(A2)}$$

We illustrate here that the pure expectations hypothesis does not hold for the Vasicek model in discrete time, in the sense that expected holding period returns are not equal for bonds of different maturities. The annualised expected holding period return of a bond with T years to maturity over a period λ beginning at time $S<T-\lambda$ is given by:

$$hpr_S(T,\lambda)=\tfrac{1}{\lambda}\log E_0\left[\frac{P(T-\lambda,r_{S+\lambda})}{P(T,r_S)}\right] \qquad \text{(A3)}$$

From (3),

$$
\begin{aligned}
hpr_S(T,\lambda) &= \tfrac{1}{\lambda}\log E_0[\frac{C(T-\lambda)}{C(T)} e^{-B(T-\lambda)r_{S+\lambda}+B(T)r_S}] \\
&= \tfrac{1}{\lambda}\log \frac{C(T-\lambda)}{C(T)} + \tfrac{1}{\lambda}\log E_0[E_S[e^{B(T)r_S-(r_S e^{-\lambda a}+b(1-e^{-\lambda a}))B(T-\lambda)-B(T-\lambda)Z_1}]] \\
&= \tfrac{1}{\lambda}\log \frac{C(T-\lambda)}{C(T)} - \tfrac{b}{\lambda}(1-e^{-\lambda a})B(T-\lambda) + B(T-\lambda)^2 \tfrac{\sigma_r^2}{2\lambda a}(1-e^{-2\lambda a}) \\
&\qquad + \tfrac{1}{\lambda}\log E_0[e^{r_S(B(T)-e^{-\lambda a}B(T-\lambda))}] \\
&= \tfrac{1}{\lambda}\log \frac{C(T-\lambda)}{C(T)} - \tfrac{b}{\lambda}(1-e^{-\lambda a})B(T-\lambda) + B(T-\lambda)^2 \tfrac{\sigma_r^2}{2\lambda a}(1-e^{-2\lambda a}) \\
&\qquad + \tfrac{1}{\lambda} r_0(B(T)-e^{-\lambda a}B(T-\lambda))e^{-Sa} + \tfrac{b}{\lambda}(1-e^{-Sa}) \\
&\qquad\qquad + \tfrac{1}{\lambda}(B(T)-e^{-\lambda a}B(T-\lambda))^2 \tfrac{\sigma_r^2}{2a}(1-e^{-2Sa}),
\end{aligned}
\tag{A4}
$$

where the last two steps use the joint conditional distribution of $r_{S+\lambda}$ and r_S conditional on r_0 given by (2), and standard results for the normal distribution. The application of L'Hopital's rule verifies that $\lim_{\lambda \to 0} hpr_0(T,\lambda) = r_0$, independent of the value of *T*. So the instantaneous expected return on bonds is independent of their term and the pure expectations hypothesis holds.

However, in our model, we only allow investors to alter their portfolios at yearly intervals. It is easy to verify that $hpr_0(T,1)$ is a decreasing function of *T*, meaning that holding short-term bonds for one year is expected to outperform longer term bonds, if only by a small amount. As *S* becomes longer, the magnitude of the effect increases. So by assuming some reinvestment risk and following a strategy of rolling over fixed-interest assets with a duration shorter than the liabilities, the pension fund can obtain a higher expected return than if it matched assets and liabilities perfectly. It is this convexity adjustment which drives a great deal of the interest rate phenomena observed in our results.

Adjusting the Brennan and Xia (2000) model to incorporate liabilities

Brennan and Xia (2000) assume that an agent dynamically chooses an investment policy in stocks and bonds to maximise the expected CRRA utility of terminal wealth, in the presence of stochastic interest rates which follow a Vasicek model. We can use a change of numeraire to adjust their model by making investors maximise the CRRA utility of terminal wealth relative to a liability, if we assume that the liability is perfectly matched by a zero-coupon bond with duration T and equities are uncorrelated with the yield curve.

This yields:

$$\mathbf{x} = \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{\pi} + (1-\frac{1}{\gamma})\Sigma^{-1}\boldsymbol{\sigma}_{\mathbf{L}}, \tag{A5}$$

where $\mathbf{x}$ is the vector of optimal portfolio weights in risky assets, in our case the stock and the long bond, $\boldsymbol{\pi}$ is the vector of risk premiums, in our case $\boldsymbol{\pi} = (\mu_e, 0)'$, Σ is the instantaneous variance-covariance matrix of the returns of the two risky assets the investor can invest in, so:

$$\Sigma = \begin{pmatrix} \sigma_S^2 & 0 \\ 0 & \sigma_{P_L}^2 \end{pmatrix}, \tag{A6}$$

and $\boldsymbol{\sigma}_{\mathbf{L}}$ is the vector of instantaneous covariances between the three assets the investor can invest in and a bond which perfectly hedges the liability, so $\boldsymbol{\sigma}_{\mathbf{L}} = (0, \sigma_{P_L L})'$. If we assume that the liability can be hedged by a zero-coupon bond with term *L* in the first period, and we assume the Vasicek model of interest rates then we can write $\sigma_{P_L L} = \sigma_r^2 B(20-t)B(L-t)$, where the formula for $B(s)$ is given in (A1). Using the same notation, $\sigma_{P_L}^2 = \sigma_r^2 B(20-t)B(20-t)$.

Numerical techniques used to solve the model

This section describes the numerical techniques used to solve the model. The trustee's decision problem at time period *t* is given by equations (9), (8), (7), (6), (4) and (2) in the main text. This is a dynamic portfolio problem with three state variables, plus time.

Because only the ratio of assets to liabilities enters into the final objective function, it is tempting to conclude that state variable reduction techniques could be used to reduce the dimensionality of the problem. However, this is not the case, because of the particular interest rate and mortality assumptions that we chose.

In the Vasicek model, the current instantaneous short rate affects the mean value of interest rate shocks, meaning that it was not possible to eliminate the instantaneous short rate as a state variable.

In the case of the mortality index parameter, the updating equation for the value of the pension liability is given by:

$$L_{t+1} - L_t = \sum_{i=9-t}^{44-t} P(i+1, r_{t+1}) \left(\prod_{j=0}^{i} (1-\exp(a_{65+j} + b_{65+j}(k_{t+1} + j\mu_k))) \right) - \sum_{i=10-t}^{45-t} P(i+1, r_t) \left(\prod_{j=0}^{i} (1-\exp(a_{65+j} + b_{65+j}(k_t + j\mu_k))) \right). \quad \text{(A7)}$$

In this double sum of exponential terms, unless the value of k_t is known, the effect of a given shock to the value of k_t on the size of pension liabilities cannot be predicted (if k_t is already large, a given shock will have a different effect than if k_t is already small).

Although we considered reducing the complexity of the model by selecting an alternative model for mortality and interest rates, in our opinion, the disadvantages of such an approach – in particular the risk of mis-specifying the covariance structure of the survival probabilities at different ages, but also potential difficulties of calibration and interpretation – outweighed the advantages.

We therefore chose to retain the Lee-Carter model and a specification of the yield curve and solved the model with three state variables, plus time. We solved the problem numerically by backward induction using techniques of stochastic dynamic programming, based on the work of Carroll (2000).

First, we discretised the state space in each state variable, using a static grid. To ensure sufficient accuracy without performing unnecessary computation, we increased the density of the grid in regions of interest – so when assets are small, around the point when the pension plan is fully funded, and around the expected long-run unconditional mean of the instantaneous short rate.

We calculated the value function in the final period at each point in the state space grid using (8). Moving one period back, for a given asset mix, we used (2), (4), (6) and (7) to calculate the expected value of the value function in the next period using Gauss-Hermite integration of a very high order over all three sources of uncertainty (equity returns, shocks to the instantaneous short rate, and mortality shocks) to ensure that the area around the kink was calculated accurately. To approximate the value function between grid points – necessary for our integration procedure – we used three-dimensional linear interpolation of a non-linear transformation of the value function described by Carroll (1997) which was found to work well for CRRA-type preferences such as the type we have here. At the boundaries of the state space grid we did not extrapolate values, to prevent unexpected sources of error arising, but we ensured that the state space was sufficiently large to prevent the propagation of errors from this approximation.

The first order condition in the decision variables for this problem is not well defined because of the discontinuity in the value function. This leads to the possibility of multiple – or no – solutions to the first order condition. In the case of multiple solutions, only one of the local maxima will be a global maximum. After experimenting with different approaches, we decided to solve the problem directly by maximising the value function itself by grid search over all possible asset mixes rather than by solving the first order condition.

We repeated the calculation each time period using backward induction. We implemented the solution in C++ on a cluster computer maintained by the High Performance Computing Centre at Imperial College.

9. Tables

Table 1: Case 1, proportion of portfolio invested in stocks, instantaneous short rate equals long-term mean.

	Proportion of portfolio invested in equities			
	NUMERICAL MODEL			BENCHMARK*
	Time period			Time period
	T-10	T-5	T-1	All
Base	0.20	0.20	0.20	0.200
No mortality uncertainty	0.20	0.20	0.20	0.200
High mortality uncertainty	0.20	0.20	0.20	0.200
ERP = 0.03	0.14	0.14	0.14	0.143
ERP = 0.05	0.25	0.25	0.25	0.250
CRRA = 3	0.33	0.33	0.33	0.333
CRRA = 7	0.14	0.14	0.14	0.143

** Benchmark makes no allowance for unhedgeable liabilities.*

Table 2: Case 1, duration of bond portfolio at different time periods, instantaneous short rate equals long-term mean.

	Duration of bond portfolio					
	NUMERICAL MODEL			BENCHMARK*		
	Time period			Time period		
	T-10	T-5	T-1	T-10	T-5	T-1
Base	15.49	12.03	9.50	20.00	15.00	11.00
No mortality uncertainty	15.49	12.03	9.50	20.00	15.00	11.00
High mortality uncertainty	15.49	12.03	9.50	20.00	15.00	11.00
ERP = 0.03	15.58	11.91	9.02	18.88	14.18	10.41
ERP = 0.05	15.69	12.20	9.93	21.27	15.93	11.67
CRRA = 3	12.34	10.19	9.06	20.00	15.00	11.00
CRRA = 7	16.69	12.72	9.60	20.00	15.00	11.00

** Benchmark makes no allowance for unhedgeable liabilities.*

Table 3: Case 2, base-case proportion of portfolio invested in stocks, different time horizons, base-case mortality uncertainty, instantaneous short rate equals long-term mean.

Expected funding ratio	Proportion of portfolio invested in equities									
	Time period									
	T-10	T-9	T-8	T-7	T-6	T-5	T-4	T-3	T-2	T-1
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.89	0.91	0.95	0.96	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.78	0.79	0.81	0.84	0.89	0.93	1.00	0.99	1.00	1.00
1.2	0.68	0.70	0.71	0.72	0.73	0.76	0.79	0.80	0.83	0.59
1.3	0.60	0.60	0.60	0.60	0.61	0.61	0.59	0.55	0.47	0.34
1.4	0.51	0.50	0.51	0.52	0.49	0.47	0.46	0.41	0.34	0.34
1.5	0.45	0.45	0.44	0.42	0.41	0.40	0.38	0.34	0.33	0.33

Table 4: Case 2, proportion of portfolio invested in stocks, different levels of equity risk premium and risk aversion of fiduciaries, base-case mortality uncertainty, instantaneous short rate equals long-term mean.

Expected	ERP = 0.03			ERP = 0.05			CRRA = 3			CRRA = 7		
funding	Time period			Time period			Time period			Time period		
ratio	T-10	T-5	T-1	T-10	T-5	T-1	T-10	T-5	T-1	T-10	T-5	T-1
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.97
0.5	1.00	1.00	0.84	1.00	1.00	1.00	1.00	1.00	1.00	0.98	1.00	0.80
0.6	1.00	1.00	0.79	1.00	1.00	1.00	1.00	1.00	1.00	0.93	1.00	0.70
0.7	1.00	1.00	0.98	1.00	1.00	1.00	1.00	1.00	1.00	0.86	1.00	0.90
0.8	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.79	0.99	1.00
0.9	0.88	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.70	0.91	1.00
1	0.75	0.99	1.00	0.96	1.00	1.00	1.00	1.00	1.00	0.62	0.79	1.00
1.1	0.63	0.82	0.94	0.85	1.00	1.00	1.00	1.00	1.00	0.54	0.65	0.94
1.2	0.54	0.64	0.27	0.77	0.88	0.67	1.00	1.00	1.00	0.45	0.47	0.26
1.3	0.45	0.44	0.26	0.69	0.73	0.44	0.97	1.00	0.92	0.38	0.36	0.25
1.4	0.39	0.34	0.25	0.62	0.60	0.41	0.90	0.93	0.57	0.33	0.28	0.24
1.5	0.33	0.28	0.24	0.56	0.53	0.41	0.82	0.81	0.56	0.28	0.24	0.23

Table 5: Case 2, duration of bond portfolio, different time horizons, base-case mortality uncertainty, instantaneous short rate equals long-term mean.

Expected funding ratio	Duration of bond portfolio									
	Time period									
	T-10	T-9	T-8	T-7	T-6	T-5	T-4	T-3	T-2	T-1
0.8	-	-	-	-	-	-	-	-	-	-
0.9	-	-	-	-	-	-	-	-	-	-
1	1.78	-	-	-	-	-	-	-	-	-
1.1	1.17	1.63	1.85	2.03	1.48	-	-	-	-	-
1.2	1.51	1.00	1.00	1.00	1.55	1.00	1.62	1.15	1.65	8.85
1.3	1.00	1.42	1.42	1.34	1.13	1.26	1.30	1.27	3.67	8.88
1.4	1.00	1.16	1.16	1.00	1.29	1.26	1.42	3.69	8.57	8.99
1.5	1.06	1.06	1.02	1.21	1.89	2.98	5.09	7.73	9.03	8.95

Table 6: Case 2, duration of bond portfolio, different levels of equity risk premium and risk aversion, base-case mortality uncertainty, instantaneous short rate equals long-term mean.

Expected funding ratio	ERP = 0.03			ERP = 0.05			CRRA = 3			CRRA = 7		
	Time period			Time period			Time period			Time period		
	T-10	T-5	T-1	T-10	T-5	T-1	T-10	T-5	T-1	T-10	T-5	T-1
0.4	-	-	-	-	-	-	-	-	-	-	-	-
0.5	-	-	1.48	-	-	-	-	-	-	-	-	3.02
0.6	-	-	1.15	-	-	-	-	-	-	-	-	5.97
0.7	-	-	-	-	-	-	-	-	-	1.56	-	9.34
0.8	-	-	-	-	-	-	-	-	-	1.22	-	-
0.9	1.00	-	-	-	-	-	-	-	-	1.57	-	-
1	1.34	-	-	-	-	-	-	-	-	1.23	1.11	-
1.1	1.51	1.00	-	2.00	-	-	-	-	-	1.08	1.13	-
1.2	1.06	1.00	7.78	1.12	1.00	10.23	-	-	-	1.35	1.26	9.23
1.3	1.35	1.18	7.92	1.31	1.38	10.11	-	-	-	2.21	2.48	9.27
1.4	1.05	1.66	8.06	1.42	1.35	10.15	1.31	-	7.66	5.62	7.77	9.30
1.5	1.51	5.71	8.09	1.36	1.52	10.16	1.87	1.18	7.84	9.27	10.37	9.31

Table 7: Case 2: portfolio composition, different levels of mortality uncertainty, instantaneous short rate equals long-term mean.

	LOW MORTALITY UNCERTAINTY						HIGH MORTALITY UNCERTAINTY					
Expected	EQUITY PROPORTION			BOND DURATION			EQUITY PROPORTION			BOND DURATION		
funding	Time period			Time period			Time period			Time period		
ratio	T-10	T-5	T-1	T-10	T-5	T-1	T-10	T-5	T-1	T-10	T-5	T-1
0.2	1.00	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
0.3	1.00	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
0.4	1.00	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
0.5	1.00	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
0.6	1.00	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
0.7	1.00	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
0.8	1.00	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
0.9	0.99	1.00	1.00	-	-	-	1.00	1.00	1.00	-	-	-
1	0.88	1.00	1.00	1.85	-	-	0.89	1.00	1.00	1.52	-	-
1.1	0.78	0.94	1.00	1.10	-	-	0.79	0.94	1.00	1.34	-	-
1.2	0.68	0.76	0.58	1.45	1.14	8.85	0.68	0.77	0.61	1.57	1.00	8.81
1.3	0.59	0.60	0.34	1.00	1.35	8.88	0.61	0.62	0.34	1.00	1.22	8.88
1.4	0.50	0.46	0.34	1.00	1.26	8.99	0.52	0.51	0.34	1.00	1.00	8.99
1.5	0.45	0.40	0.33	1.10	3.30	8.96	0.46	0.42	0.33	1.00	2.49	8.95

Table 8: Case 1 and case 2, Monte-Carlo simulations of funding ratio, based on optimal investment strategies.

	CASE 1: FUNDING RATIO										
	Time period										
Percentile	T-10	T-9	T-8	T-7	T-6	T-5	T-4	T-3	T-2	T-1	T
0.01	0.80	0.73	0.70	0.68	0.67	0.65	0.64	0.63	0.63	0.62	0.62
0.05	0.80	0.75	0.73	0.72	0.71	0.70	0.69	0.69	0.68	0.68	0.68
0.25	0.80	0.78	0.78	0.77	0.77	0.77	0.77	0.78	0.78	0.78	0.78
0.5	0.80	0.80	0.81	0.82	0.82	0.83	0.83	0.84	0.85	0.85	0.86
0.75	0.80	0.83	0.85	0.86	0.87	0.89	0.90	0.91	0.92	0.93	0.95
0.95	0.80	0.87	0.91	0.94	0.96	0.99	1.01	1.04	1.05	1.08	1.10
0.99	0.80	0.91	0.96	1.00	1.03	1.07	1.09	1.13	1.17	1.20	1.23

	CASE 2: FUNDING RATIO										
	Time period										
Percentile	T-10	T-9	T-8	T-7	T-6	T-5	T-4	T-3	T-2	T-1	T
0.01	0.80	0.51	0.43	0.38	0.34	0.31	0.29	0.26	0.25	0.24	0.23
0.05	0.80	0.59	0.52	0.48	0.45	0.43	0.41	0.39	0.37	0.36	0.36
0.25	0.80	0.71	0.69	0.67	0.67	0.66	0.66	0.66	0.66	0.65	0.66
0.5	0.80	0.81	0.84	0.86	0.88	0.90	0.93	0.96	0.99	1.04	1.08
0.75	0.80	0.93	1.01	1.07	1.14	1.20	1.27	1.33	1.41	1.46	1.50
0.95	0.80	1.13	1.29	1.41	1.52	1.62	1.72	1.78	1.84	1.89	1.95
0.99	0.80	1.29	1.51	1.68	1.79	1.91	2.00	2.08	2.16	2.24	2.34

10. Figures

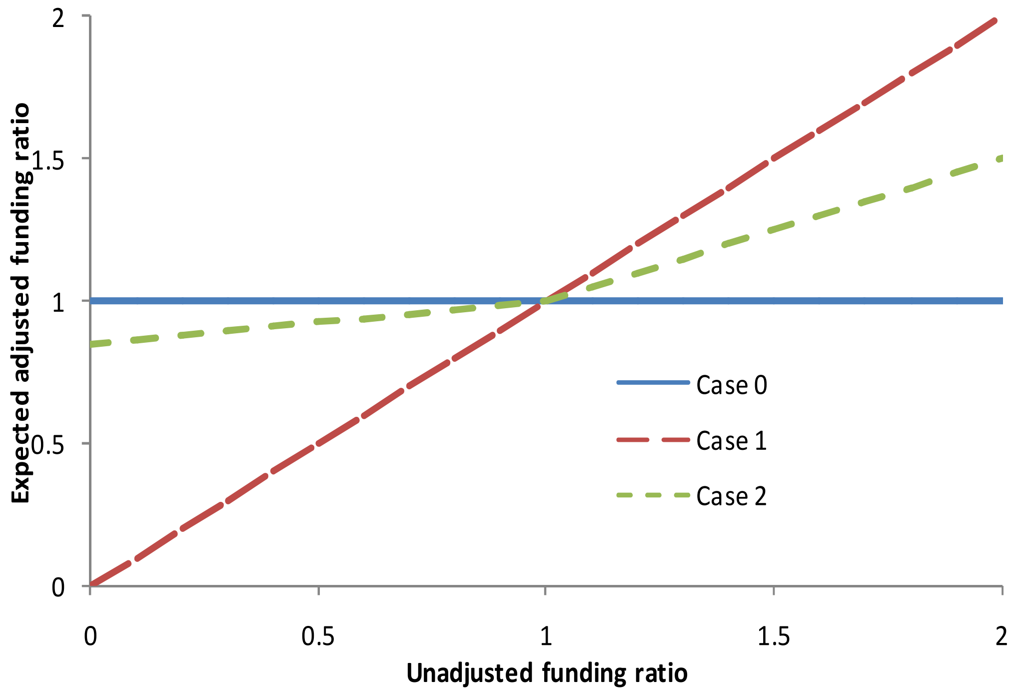

Figure 1: Expected adjusted funding ratio as a function of the unadjusted funding ratio at the terminal date, cases 0, 1 and 2.

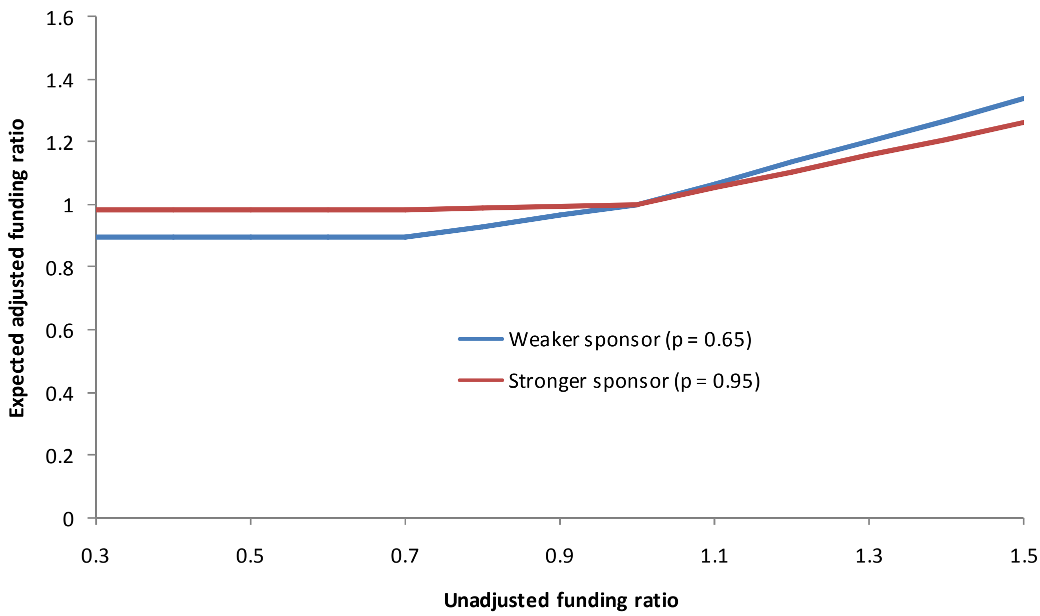

Figure 2: Expected adjusted funding ratio as a function of the unadjusted funding ratio with two kinks.